

Differential Structure of Abelian Functions

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Abstract

The space of abelian functions of a principally polarized abelian variety (J, Θ) is studied as a module over the ring \mathcal{D} of global holomorphic differential operators on J . We construct a \mathcal{D} free resolution in case Θ is non-singular. As an application, in the case of dimension 2 and 3, we construct a new linear basis of the space of abelian functions which are singular only on Θ in terms of logarithmic derivatives of the higher dimensional σ -function.

1 Introduction

Let (J, Θ) be a g -dimensional principally polarized Abelian variety and A the affine ring of $J - \Theta$. We express J as the quotient of the g -dimensional vector space by some lattice, $J = \mathbb{C}^g / \Gamma$ and Θ as the zero locus of a theta function $\theta(z)$ with $z = (z_1, \dots, z_g)$ being linear coordinates of \mathbb{C}^g . Analytically A is isomorphic to the ring of meromorphic functions on J which have poles only on Θ . Such functions can be considered as meromorphic and periodic functions on \mathbb{C}^g which have poles only on $(\theta(z) = 0)$. Obviously if we differentiate such a function with respect to z_i then we again get a function with the same property. This means that A becomes a module over the ring of differential operators $\mathcal{D} = \mathbb{C}[\partial_1, \dots, \partial_g]$, $\partial_i = \partial / \partial z_i$. It is a very curious problem to determine generators and relations of the \mathcal{D} -module A . The aim of this paper is to study these problems for (J, Θ) with Θ being non-singular.

The case of dimension one is known from the classical theory of elliptic functions. In this case the structure of A is very simple. Let $\wp(z)$ be the Weierstrass elliptic function and p the point of the elliptic curve corresponding to $z = 0$. In this case $\mathcal{D} = \mathbb{C}[\partial]$, $\partial = \frac{d}{dz}$. As a \mathcal{D} -module A is generated by 1 and $\wp(z)$. More precisely 1, $\wp(z)$, $\wp'(z)$, $\wp''(z)$, ... give a \mathbb{C} -linear basis of A , where $\wp'(z) = \frac{d}{dz}\wp(z)$ etc. This fact is incorporated in the beautiful addition formula of Frobenius and Stickelberger:

$$(-1)^{\frac{(n-1)(n-2)}{2}} \prod_{k=1}^{n-1} k! \frac{\sigma(z_1 + \dots + z_n) \prod_{i < j} \sigma(z_i - z_j)}{\prod_{j=1}^n \sigma(z_j)^n} = \begin{vmatrix} 1 & \dots & 1 \\ \wp(z_1) & \dots & \wp(z_n) \\ \vdots & & \vdots \\ \wp^{(n-2)}(z_1) & \dots & \wp^{(n-2)}(z_n) \end{vmatrix}, \quad (1)$$

where $\sigma(z)$ is the Weierstrass sigma function. Consider both hand sides of (1) as a function of z_1 . Then the left hand side is an element of A whose order of poles at p is at most n . The right hand side of (1) expresses the left hand side as a linear combination of the basis 1, $\wp'(z)$, ..., $\wp^{(n-2)}(z)$.

Up to now not many is known for the \mathcal{D} -module structure of A in the case of higher dimensions. In [9] the case of hyperelliptic Jacobians is studied and a conjecture on the \mathcal{D} -free resolution of A is given. Up to now it is still difficult to prove the conjecture in general. Moreover few is known on the \mathcal{D} -module

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structure of A for non-hyperelliptic Jacobians. In the present paper we construct a \mathcal{D} -free resolution of A in the generic case which means that Θ is non-singular. We remark that in this generic case similar problem for non-trivial flat line bundles on J is studied in [7].

The content of the paper is as follows. In section 2 the \mathcal{D} -module structure of the affine ring of an abelian variety and its relation to the algebraic de Rham complex are explained. Large degree components of the highest cohomology group of the graded de Rham complex are studied in section 3. As a consequence the affine ring A is proved to be a finitely generated \mathcal{D} -module here. In section 4 small degree components of the highest cohomology group are studied. The dimension of each homogeneous component of the highest cohomology group are determined here. In section 5 to 7 characters of cohomology groups, affine ring and some related symplectic vector space are calculated. A \mathcal{D} -free resolution is constructed in section 8. In section 9 the results of §8 is interpreted into the term of theta functions. As examples a linear basis of A is given in terms of logarithmic derivatives of a theta function in the case of genus two and three. Three appendices provide proofs of Lemmas and assertions which are used in the main body of the paper.

2 Affine ring

Let (J, Θ) be a principally polarized abelian variety of dimension g . Throughout this paper we assume that $g \geq 2$ and Θ is non-singular. This means, in particular, that Jacobian varieties of hyperelliptic curves of genus $g \geq 3$ and non-hyperelliptic curves of genus $g \geq 4$ are excluded.

Let \mathcal{O} be the sheaf of germs of holomorphic functions on J , $\mathcal{O}(n)$ ($n \geq 0$) the sheaf of germs of meromorphic functions on J which have poles only on Θ of order at most n and $\mathcal{O}(*)$ the sheaf of germs of meromorphic functions on J which have poles only on Θ . We set $A = H^0(J, \mathcal{O}(*))$. It is isomorphic to the affine coordinate ring of $J - \Theta$. The ring A has an increasing filtration determined by the order of poles on Θ ,

$$A = \cup_{n=0}^{\infty} A_n, \quad A_n = H^0(J, \mathcal{O}(n)).$$

We set $A_n = 0$ for $n < 0$ for convenience.

Analytically A is described in the following manner. Let τ be a g by g symmetric matrix whose imaginary part is positive definite, $J = \mathbb{C}/\mathbb{Z}^g + \tau\mathbb{Z}^g$ and $\theta(z)$ the Riemann's theta function,

$$\theta(z) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i {}^t n \tau n + 2\pi i {}^t n z), \quad z = (z_1, \dots, z_g).$$

The theta divisor is defined by $\Theta = (\theta(z) = 0) \subset J$. Then

$$A_n = \left\{ \frac{f(z)}{\theta(z)^n} \right\},$$

where $f(z)$ runs over all holomorphic functions on \mathbb{C}^g with the property

$$\frac{f(z + \tau p + q)}{\theta(z + \tau p + q)^n} = \frac{f(z)}{\theta(z)^n} \quad (2)$$

for any $p, q \in \mathbb{Z}^g$. Notice that the relation (2) is preserved by the differentiation with respect to z_i . Thus A becomes a module over the ring of differential operators $\mathcal{D} = \mathbb{C}[\partial_1, \dots, \partial_g]$ where $\partial_i = \partial/\partial z_i$.

In order to study the \mathcal{D} -module structure of A it is convenient to consider the garded ring $\text{gr } A$ associated with the filtration of A ,

$$\text{gr } A = \oplus_{n=0}^{\infty} \text{gr}_n A, \quad \text{gr}_n A = A_n / A_{n-1}.$$

Since the action of ∂_i on A satisfies the relation

$$\partial_i A_n \subset A_{n+1},$$

$\text{gr } A$ also becomes a \mathcal{D} -module. Thus it is possible to define the deRham complex associated with $\text{gr } A$ as follows. Let $T^* = \sum_{i=1}^g \mathbb{C} dz_i$ be the space of translation invariant holomorphic one forms on J . Define the map

$$d : \text{gr } A \otimes \wedge^p T^* \longrightarrow \text{gr } A \otimes \wedge^{p+1} T^*,$$

by $d = \sum_{i=1}^g \partial_i \otimes dz_i$. It obviously defines a complex $(\text{gr } A \otimes \wedge^\bullet T^*, d)$. We denote by H^k its k -th cohomology group $H^k(\text{gr } A \otimes \wedge^\bullet T^*)$. Notice that

$$H^g \simeq \text{gr } A / \sum_{i=1}^g \partial_i \text{gr } A,$$

as a vector space. A basis of this space gives a minimal set of generators of $\text{gr } A$ as a \mathcal{D} -module. Therefore we have to study the cohomology groups of $(\text{gr } A \otimes \wedge^\bullet T^*, d)$. To this end we introduce a grading on H^k as follows. Let us assign degree to elements in $\wedge^p T^*$ by

$$\deg dz_{i_1} \wedge \cdots \wedge dz_{i_g} = -p.$$

The tensor product $\text{gr } A \otimes \wedge^p T^*$ of two graded spaces is naturally graded. Since the map d preserves the grading, H^k becomes graded. Let

$$H^k = \oplus_{n \geq -k} H_n^k$$

be the decomposition into homogeneous components.

3 Cohomology of large degree

We first study the case of n large.

Proposition 1 *The following isomorphisms holds.*

$$H_n^g \simeq \begin{cases} 0, & n \geq 2 \\ H^g(J, \mathcal{O}), & n = 1. \end{cases}$$

We set

$$dz^g = dz_1 \wedge \cdots \wedge dz_g.$$

An element of H_n^g can be written in the form $f dz^g$ for some $f \in \text{gr}_{n+g} A$. We denote the vector space of such f 's by $H_n^g(dz^g)^{-1}$.

Corollary 1 *As a \mathcal{D} -module, $\text{gr } A$ is generated by $\oplus_{n=-g}^1 H_n^g(dz^g)^{-1}$.*

For the proof of Proposition 1 we need to introduce some notations on sheaves. Let Ω^p be the sheaf of germs of holomorphic p -forms on J , $\Omega^p(n)$ ($n \geq 0$) the sheaf of germs of meromorphic p -forms on J which have poles only on Θ of order at most n , $\Omega^p(-n)$ ($n \geq 0$) the sheaf of germs of holomorphic p -forms on J which have zeros on Θ of order at least n and $\text{gr}_n \Omega^p = \Omega^p(n) / \Omega^p(n-1)$.

Since Ω^p is a free \mathcal{O}_J -module, the following relations are valid

$$\mathrm{gr}_n \Omega^p \simeq \mathrm{gr}_n \mathcal{O} \otimes \Omega^p, \quad (3)$$

$$H^k(J, \mathrm{gr}_n \Omega^p) \simeq H^k(J, \mathrm{gr}_n \mathcal{O}) \otimes H^0(J, \Omega^p). \quad (4)$$

$$H^k(J, \Omega^p(n)) \simeq H^k(J, \mathcal{O}(n)) \otimes H^0(J, \Omega^p). \quad (5)$$

To study cohomology groups of those sheaves the following vanishing property of cohomologies due to Mumford [5] is important.

Lemma 1 [5] *We have*

$$H^k(J, \mathcal{O}(n)) = 0, \quad k \geq 1, n \geq 1. \quad (6)$$

The next lemma easily follows from this.

Lemma 2

$$H^i(J, \mathrm{gr}_n \mathcal{O}) \simeq \begin{cases} 0, & n < 0, \quad i \leq g-2, \\ H^i(J, \mathcal{O}), & n = 0, \quad i \leq g-2, \\ H^{i+1}(J, \mathcal{O}), & n = 1, \quad i \geq 0, \\ 0, & n > 1, \quad i \geq 1 \end{cases}$$

Notice that $H^0(J, \Omega^p) \simeq \wedge^p H^0(J, \Omega^1) = \wedge^p T^*$. Using (4), (5) and Lemma 1 we have

$$H^0(J, \Omega^p(n)) \simeq A_n \otimes \wedge^p T^*, \quad (7)$$

$$H^0(J, \mathrm{gr}_n \Omega^p) \simeq \mathrm{gr}_n A \otimes \wedge^p T^*, \quad n \geq 2. \quad (8)$$

The exterior differentiation defines a map $d : \Omega^p(n) \longrightarrow \Omega^{p+1}(n+1)$ which induces a map $d : \mathrm{gr}_n \Omega^p \longrightarrow \mathrm{gr}_{n+1} \Omega^{p+1}$. The induced map $d : H^0(J, \mathrm{gr}_n \Omega^p) \longrightarrow H^0(J, \mathrm{gr}_{n+1} \Omega^{p+1})$ on the cohomology groups is the same as that of the complex $(\mathrm{gr} A \otimes \wedge^\bullet, d)$ due to the isomorphism (8).

Define the sheaf Φ_n^p , $n \geq 1$ as the kernel of the map $d : \Omega^p(n) \longrightarrow \Omega^{p+1}(n+1)$. By the definition the following sequence is exact

$$0 \longrightarrow \Phi_n^p \longrightarrow \mathrm{gr}_n \Omega^p \xrightarrow{d} d\mathrm{gr}_n \Omega^p \longrightarrow 0.$$

Notice that the map $d : \mathrm{gr}_n \Omega^p \longrightarrow \mathrm{gr}_{n+1} \Omega^{p+1}$ is \mathcal{O} linear and the sheaf Φ_n^p , $d\mathrm{gr}_n \Omega^p$ become a coherent \mathcal{O} -module. Let us set $\Xi = d \log \theta$. It defines a map

$$\Xi \wedge : \mathrm{gr}_n \Omega^p \longrightarrow \mathrm{gr}_{n+1} \Omega^{p+1}.$$

Lemma 3 (i) *For $p \geq 1$*

$$\Phi_n^p \simeq \begin{cases} d\mathrm{gr}_{n-1} \Omega^{p-1}, & n \geq 2, \\ \Xi \wedge \mathrm{gr}_0 \Omega^{p-1}, & n = 1 \end{cases}$$

(ii) $\Xi \wedge \mathrm{gr}_n \mathcal{O} \simeq \mathrm{gr}_n \mathcal{O}$ for any integer n .

(iii) $\mathrm{gr}_n \mathcal{O} \simeq d\mathrm{gr}_n \mathcal{O}$.

(iv) $\mathrm{Ker}(\Xi \wedge : \mathrm{gr}_{-n} \Omega^p \longrightarrow \mathrm{gr}_{-n+1} \Omega^{p+1}) \simeq \Xi \wedge \mathrm{gr}_{-n-1} \Omega^{p-1}$, $p \geq 1$, $n \geq 0$.

Proof. Since every statement can be proved in a similar way, we shall give a proof of (i). It is obvious that the support of Φ_n^p is contained in Θ . Let Q be a point of Θ and (z_0, \dots, z_{g-1}) be a local coordinate system around Q such that $z_0 = 0$ is a local defining equation of Θ . Write a local section of $\text{gr}_n \Omega^p$ as

$$\eta = \frac{1}{z_0^n} dz_0 \wedge \eta_1 + \frac{1}{z_0^n} \eta_2,$$

where η_1, η_2 does not contain dz_0 and z_0 . Then, in $\text{gr}_{n+1} \Omega^p$,

$$d\eta = -\frac{n}{z_0^{n+1}} \wedge \eta_2.$$

Thus $d\eta = 0$ in $\text{gr}_{n+1} \Omega^p$ is equivalent to $\eta_2 = 0$. Therefore η is a local section of Φ^p if and only if it is written as $\eta = \frac{1}{z_0^n} dz_0 \wedge \eta_1$ with η_1 astisfying the condition above. If $n \geq 2$, then $\eta = \frac{1}{1-n} d\left(\frac{\eta_1}{z_0^{n-1}}\right)$ in $\text{gr}_n \Omega^p$ and $\Phi_n^p = d\text{gr}_{n-1} \Omega^{p-1}$. If $n = 1$, then $\eta = \frac{dz_0}{z_0} \wedge \eta_1$. Thus $\Phi_1^p = d \log \theta \wedge \text{gr}_0 \Omega^{p-1}$. ■

Proof of Proposition 1.

By the definition

$$H_{n-g}^g = \frac{\text{gr}_n A \otimes \wedge^g T^*}{d(\text{gr}_{n-1} A \otimes \wedge^{g-1} T^*)}. \quad (9)$$

By (8) it is sufficient to prove

$$\frac{H^0(J, \text{gr}_n \Omega^g)}{dH^0(J, \text{gr}_{n-1} \Omega^{g-1})} \simeq \begin{cases} 0, & n \geq g+2 \\ H^g(J, \mathcal{O}), & n = g+1. \end{cases}$$

Since $H^i(J, \text{gr}_n \Omega^p) = 0$ for $n \geq 2, i \geq 1$ by Lemma 2 and (4), the cohomology sequence of

$$0 \longrightarrow d\text{gr}_{n-2} \Omega^{g-2} \longrightarrow \text{gr}_{n-1} \Omega^{g-1} \xrightarrow{d} \text{gr}_n \Omega^g \longrightarrow 0, \quad n \geq 3, \quad (10)$$

gives the isomorphism

$$\frac{H^0(J, \text{gr}_n \Omega^g)}{dH^0(J, \text{gr}_{n-1} \Omega^{g-1})} \simeq H^1(J, d\text{gr}_{n-2} \Omega^{g-2}), \quad n \geq 3. \quad (11)$$

Using similar sheaf exact sequences we get, for $n \geq g+1$,

$$H^1(J, d\text{gr}_{n-2} \Omega^{g-2}) \simeq H^2(J, d\text{gr}_{n-3} \Omega^{g-3}) \simeq \dots \simeq H^{g-1}(J, d\text{gr}_{n-g} \mathcal{O}).$$

The last cohomology can be easily calculated as desired using Lemma 3 (iii) and Lemma 2. ■

4 Cohomology of small degree

Next we study H_n^g for small n . Let us set

$$a_n^{(g)} = \dim H_{n-g}^g.$$

Obviously $a_0^{(g)} = 1$, $a_1^{(g)} = 0$ and $a_2^{(g)} = 2g - 1$. By Proposition 1 we have $a_{g+1}^{(g)} = 1$ and $a_n^{(g)} = 0$ for $n \geq g+2$. The remaining values of $a_n^{(g)}$ are given by

Proposition 2 For $2 \leq n \leq g+1$ we have

$$\begin{aligned} a_n^{(g)} &= (-1)^{n-1} \binom{g}{n-2} + \sum_{i=2}^n (-1)^{n-i} \binom{g+1}{n-i} i^g \\ &+ \sum_{i=0}^{n-3} (-1)^i \left(\binom{g}{n-1} \binom{g}{n-2-i} - \binom{g}{n} \binom{g}{n-3-i} \right). \end{aligned} \quad (12)$$

Proof. The case of $n = 2$ can be easily checked. The proof for $n = g+1$ is given in Proposition 3 (i). Therefore we assume $3 \leq n \leq g$.

By (8) and (9)

$$a_n^{(g)} = \dim H^0(J, \text{gr}_n \Omega^g) - \dim dH^0(J, \text{gr}_{n-1} \Omega^{g-1}).$$

We have

$$dH^0(J, \text{gr}_{n-1} \Omega^{g-1}) \simeq \frac{H^0(J, \text{gr}_{n-1} \Omega^{g-1})}{H^0(J, d\text{gr}_{n-2} \Omega^{g-2})}, \quad n \geq 3,$$

by the cohomology sequence of (10). The dimension of A_n is known as (cf. [5, 6])

$$\dim A_n = n^g, \quad n \geq 1, \quad (13)$$

Due to (8) we have

$$\dim H^0(J, \text{gr}_n \Omega^g) = n^g - (n-1)^g, \quad n \geq 2. \quad (14)$$

Therefore

$$\begin{aligned} a_n^{(g)} &= \dim H^0(J, \text{gr}_n \Omega^g) - \dim H^0(J, \text{gr}_{n-1} \Omega^{g-1}) + \dim H^0(J, d\text{gr}_{n-2} \Omega^{g-2}) \\ &= n^g - (g+1)(n-1)^g + g(n-2)^g + \dim H^0(J, d\text{gr}_{n-2} \Omega^{g-2}), \quad n \geq 3. \end{aligned} \quad (15)$$

Let us calculate the last term of (15).

Consider the exact sequence

$$0 \longrightarrow d\text{gr}_{k-1} \Omega^{g-n+k-1} \longrightarrow \text{gr}_k \Omega^{g-n+k} \xrightarrow{d} d\text{gr}_k \Omega^{g-n+k} \longrightarrow 0, \quad k \geq 2. \quad (16)$$

The long cohomology sequence of (16) gives the exact sequence

$$\begin{aligned} 0 &\longrightarrow H^0(J, d\text{gr}_{k-1} \Omega^{g-n+k-1}) \longrightarrow H^0(J, \text{gr}_k \Omega^{g-n+k}) \longrightarrow H^0(J, d\text{gr}_k \Omega^{g-n+k}) \\ &\longrightarrow H^1(J, d\text{gr}_{k-1} \Omega^{g-n+k-1}) \longrightarrow 0, \end{aligned} \quad (17)$$

and the isomorphisms

$$H^i(J, d\text{gr}_k \Omega^{g-n+k}) \simeq H^{i+1}(J, d\text{gr}_{k-1} \Omega^{g-n+k-1}), \quad i \geq 1. \quad (18)$$

By (17) we have

$$\begin{aligned} \dim H^0(J, d\text{gr}_k \Omega^{g-n+k}) &= -\dim H^0(J, d\text{gr}_{k-1} \Omega^{g-n+k-1}) + \dim H^0(J, \text{gr}_k \Omega^{g-n+k}) \\ &+ \dim H^1(J, d\text{gr}_{k-1} \Omega^{g-n+k-1}). \end{aligned} \quad (19)$$

Multiplying $(-1)^{n-k}$ to both hand sides of (19) and taking summation in k from 2 to $n-2$ we get

$$\begin{aligned} \dim H^0(J, d\text{gr}_{n-2} \Omega^{g-2}) &= (-1)^{n-1} \dim H^0(J, d\text{gr}_1 \Omega^{g-n+1}) + \sum_{k=2}^{n-2} (-1)^{n-k} \dim H^0(J, \text{gr}_k \Omega^{g-n+k}) \\ &\quad + \sum_{k=2}^{n-2} (-1)^{n-k} \dim H^1(J, d\text{gr}_{k-1} \Omega^{g-n+k-1}). \end{aligned} \quad (20)$$

By (8) and (14) we know that

$$\dim H^0(J, \text{gr}_k \Omega^{g-n+k}) = \binom{g}{n-k} (k^g - (k-1)^g), \quad k \geq 2. \quad (21)$$

Let us determine the remaining part in (20).

Using repeatedly the isomorphism (18) we get

$$H^1(J, d\text{gr}_{k-1} \Omega^{g-n+k-1}) \simeq H^2(J, d\text{gr}_{k-2} \Omega^{g-n+k-2}) \simeq \dots \simeq H^{k-1}(J, d\text{gr}_1 \Omega^{g-n+1}), \quad (22)$$

for $k \geq 2$. Therefore one has to calculate the dimension of $H^i(J, d\text{gr}_1 \Omega^{g-n+1})$.

By Lemma 3 (i) the following sequence is exact

$$0 \longrightarrow \Xi \wedge \text{gr}_0 \Omega^{g-n} \longrightarrow \text{gr}_1 \Omega^{g-n+1} \xrightarrow{d} d\text{gr}_1 \Omega^{g-n+1} \longrightarrow 0. \quad (23)$$

The long cohomology exact sequence of (23) is

$$\dots \longrightarrow H^i(J, \Xi \wedge \text{gr}_0 \Omega^{g-n}) \xrightarrow{\alpha} H^i(J, \text{gr}_1 \Omega^{g-n+1}) \longrightarrow H^i(J, d\text{gr}_1 \Omega^{g-n+1}) \longrightarrow \dots \quad (24)$$

Let us study the map α .

Lemma 4 *We have*

$$H^i(J, \Xi \wedge \text{gr}_{-k} \Omega^{g-n-k}) = 0, \quad 0 \leq i \leq n+k-2, \quad 1 \leq k \leq g-n.$$

Proof. We can assume $n < g$. Let us prove the lemma by descending induction on k . For $k = g-n$,

$$H^i(J, \Xi \wedge \text{gr}_{-(g-n)} \mathcal{O}) \simeq H^i(J, \text{gr}_{-(g-n)} \mathcal{O}) = 0, \quad i \leq g-2,$$

by Lemma 3 (i),(ii) and Lemma 2. Suppose that the lemma holds from $k+1$ to $g-n$.

By Lemma 3 (iv) we have the exact sequence

$$0 \longrightarrow \Xi \wedge \text{gr}_{-k-1} \Omega^{g-n-k-1} \longrightarrow \text{gr}_{-k} \Omega^{g-n-k} \xrightarrow{\Xi \wedge} \Xi \wedge \text{gr}_{-k} \Omega^{g-n-k} \longrightarrow 0,$$

for $0 \leq k \leq g-n-1$. Then

$$H^i(J, \Xi \wedge \text{gr}_{-k} \Omega^{g-n-k}) = 0, \quad 0 \leq i \leq n+k-2,$$

by the cohomology sequenec of (25), Lemma 2 and the induction hypothesis. Thus the lemma is proved. ■

Lemma 5 $H^i(J, \Xi \wedge \text{gr}_0 \Omega^{g-n}) \simeq H^i(J, \text{gr}_0 \Omega^{g-n}), \quad i \leq n-2.$

Proof. The lemma follows from the long cohomology sequence of (25), $k = 0$, and Lemma 4. ■

Let $\bar{T}^* = \oplus_{i=1}^g \mathbb{C} d\bar{z}_i$, where $\bar{}$ denotes the complex conjugation. Then $H^i(J, \mathcal{O}) \simeq \wedge^i \bar{T}^*$. By Lemma 2, Lemma 5 and (5) we have

$$H^i(J, \Xi \wedge \text{gr}_0 \Omega^{g-n}) \simeq \wedge^i \bar{T}^* \wedge^{g-n} T^*, \quad i \leq n-2, \quad (25)$$

$$H^i(J, \text{gr}_1 \Omega^{g-n+1}) \simeq \wedge^{i+1} \bar{T}^* \wedge^{g-n+1} T^*, \quad i \geq 0. \quad (26)$$

Let

$$\hat{\omega} = \pi \sum_{i,j=1}^g (\text{Im}(\tau)^{-1})_{ij} d\bar{z}_i \wedge dz_j \in \bar{T}^* \wedge T^*.$$

Lemma 6 *In the description of (25), (26) the map α is given by the wedging $\hat{\omega}$.*

The proof of this lemma is given in Appendix A.

Remark From the cohomology sequence of

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}(1) \longrightarrow d\mathcal{O}(1) \longrightarrow 0,$$

we have

$$H^1(J, \mathbb{C}) \simeq H^0(J, d\mathcal{O}(1)). \quad (27)$$

We set

$$\zeta_i(z) = \partial_i \log \theta(z). \quad (28)$$

Then the one form $d\zeta_j$ is naturally an element of the right hand side of (27) and it can be considered as an element of $H^1(J, \mathbb{C}) \simeq \bar{T}^* \oplus T^*$. Let

$$\omega = \sum_{j=1}^g d\zeta_j \wedge dz_j \quad (29)$$

be the element of $H^2(J, \mathbb{C}) \simeq \wedge^2 H^1(J, \mathbb{C})$. Then we have

$$\omega = \hat{\omega} \quad \text{in } H^2(J, \mathbb{C}). \quad (30)$$

The proof of (30) is given in Appendix B.

By the standard argument using the representation theory of sl_2 we have that the map

$$\hat{\omega} \wedge : \oplus_{i+j=n} \wedge^i \bar{T}^* \wedge^j T^* \longrightarrow \oplus_{i+j=n+2} \wedge^i \bar{T}^* \wedge^j T^*$$

is injective for $n \leq g-1$. Thus we have

Lemma 7 *The map α is injective for $0 \leq i \leq n-2$.*

By Lemma 7, (24) splits into exact sequences

$$\begin{aligned} 0 \longrightarrow H^i(J, \Xi \wedge \text{gr}_0 \Omega^{g-n}) &\xrightarrow{\alpha} H^i(J, \text{gr}_1 \Omega^{g-n+1}) \longrightarrow H^i(J, d\text{gr}_1 \Omega^{g-n+1}) \longrightarrow 0, \\ 0 \leq i \leq n-3, \end{aligned} \quad (31)$$

$$0 \longrightarrow H^{n-2}(J, \Xi \wedge \text{gr}_0 \Omega^{g-n}) \xrightarrow{\alpha} H^{n-2}(J, \text{gr}_1 \Omega^{g-n+1}) \longrightarrow H^{n-2}(J, d\text{gr}_1 \Omega^{g-n+1}) \longrightarrow \dots \quad (32)$$

From (25), (26), (31) we get

$$\begin{aligned} \dim H^i(J, d\text{gr}_1 \Omega^{g-n+1}) &= \dim \wedge^{i+1} \bar{T}^* \wedge^{g-n+1} T^* - \dim \wedge^i \bar{T}^* \wedge^{g-n} T^* \\ &= \binom{g}{i+1} \binom{g}{n-1} - \binom{g}{i} \binom{g}{n}, \quad 0 \leq i \leq n-3. \end{aligned} \quad (33)$$

We substitute (21), (33), using (22), into (20) we get

$$\begin{aligned} &\dim H^0(J, d\text{gr}_{n-2} \Omega^{g-2}) \\ &= \sum_{k=1}^{n-2} (-1)^{n-k} \dim H^{k-1}(J, d\text{gr}_1 \Omega^{g-n+1}) + \sum_{k=2}^{n-2} (-1)^{n-k} \dim H^0(J, \text{gr}_k \Omega^{g-n+k}) \\ &= \sum_{k=0}^{n-3} (-1)^{n-3-k} \left(\binom{g}{n-1} \binom{g}{k+1} - \binom{g}{n} \binom{g}{k} \right) + \sum_{k=2}^{n-2} (-1)^{n-k} \binom{g}{n-k} (k^g - (k-1)^g) \\ &= \sum_{k=0}^{n-3} (-1)^k \left(\binom{g}{n-1} \binom{g}{n-2-k} - \binom{g}{n} \binom{g}{n-3-k} \right) + \binom{g}{2} (n-2)^g \\ &\quad + (-1)^{n-1} \binom{g}{n-2} + \sum_{i=2}^{n-3} (-1)^{n-i} \binom{g+1}{n-i} i^g. \end{aligned} \quad (34)$$

Then the proposition follows from (15) and (34). ■

Proposition 3 (i) *The right hand side of (12) is equal to one.*
(ii)

$$\sum_{n=0}^{g+1} a_n^{(g)} = \binom{2g}{g} - \binom{2g}{g-2} + g! - \frac{(2g)!}{g!(g+1)!}.$$

Proof (1) We have

$$\begin{aligned} a_{g+1}^{(g)} &= (-1)^g \binom{g}{g-1} + \sum_{i=2}^{g+1} (-1)^{g+1-i} \binom{g+1}{g+1-i} i^g + \sum_{i=0}^{g-2} (-1)^i \binom{g}{g-1-i} \\ &= (-1)^{g+1} + \sum_{i=1}^{g+1} (-1)^{g+1-i} \binom{g+1}{i} i^g + \sum_{i=0}^{g-2} (-1)^i \binom{g}{i} \\ &= 1, \end{aligned}$$

where in the last equality we use

$$\sum_{i=0}^g (-1)^i \binom{g}{i} = 0, \quad 0 = \left(x \frac{d}{dx} \right)^g (1+x)^{g+1} \Big|_{x=-1} = \sum_{i=0}^{g+1} (-1)^i \binom{g+1}{i} i^g.$$

(2) Write

$$\begin{aligned}\sum_{n=2}^{g+1} a_n^{(g)} &= S_1 + S_2 + S_3, \\ S_1 &= \sum_{n=2}^{g+1} (-1)^{n-1} \binom{g}{n-2}, \\ S_2 &= \sum_{n=2}^{g+1} \sum_{i=2}^n (-1)^{n-i} \binom{g+1}{n-i} i^g, \\ S_3 &= \sum_{n=3}^{g+1} \sum_{i=0}^{n-3} (-1)^i \left(\binom{g}{n-1} \binom{g}{n-2-i} - \binom{g}{n} \binom{g}{n-3-i} \right).\end{aligned}$$

The sum S_1 is easily evaluated as

$$S_1 = (-1)^g.$$

To calculate S_3 we use

$$\sum_{n-m=k} \binom{g}{n} \binom{g}{m} = \binom{2g}{g-k},$$

which follows from

$$(1+x)^g (1+x^{-1})^g = x^{-g} (1+x)^{2g}.$$

We have

$$S_3 = -1 + \binom{2g}{g-1} - \binom{2g}{g-2}.$$

We use

$$(1-x)^{g+1} \left(x \frac{d}{dx} \right)^g (1-x)^{-1} = \sum_{i=1}^{g+1} \sum_{n=i}^{g+1} (-1)^{n-i} \binom{g+1}{n-i} i^g x^n.$$

to deduce

$$S_2 = g! + (-1)^{g+1}.$$

The assertion follows from these results. ■

It is known that ([9])

$$\dim H^g(J - \Theta, \mathbb{C}) = \binom{2g}{g} - \binom{2g}{g-2} + g! - \frac{(2g)!}{g!(g+1)!}.$$

By the algebraic de Rham theorem

$$H^i(J - \Theta, \mathbb{C}) \simeq H^i(A \otimes \wedge^i T^*, d).$$

Then the above proposition gives a relation between the highest cohomology group of the algebraic de Rham complex $(A \otimes \wedge^i T^*, d)$ and that of the graded complex $(\text{gr } A \otimes \wedge^i T^*, d)$.

Corollary 2 *There is an isomorphism*

$$H^g \simeq H^g(A \otimes \wedge^* T^*).$$

Proof. Take a set of representatives $\{\eta_i dz^g\}$, $\eta_i \in \text{gr}_{n_i} A$, of a basis of H^g . Let $f_i \in A_{n_i}$ be a representative of η_i . Then we have a surjective map

$$\begin{aligned} H^g &\longrightarrow H^g(A \otimes \wedge^* T^\bullet), \\ \eta_i dz^g &\mapsto f_i dz^g. \end{aligned}$$

By Proposition 3 this map is also injective. ■

We list some values of $a_n^{(g)}$ for small g :

$$\begin{aligned} a_0^{(2)} &= 1, & a_1^{(2)} &= 0, & a_2^{(2)} &= 3, & a_3^{(2)} &= 1, \\ \dim H^2(J - \Theta, \mathbf{C}) &= 5 = 1 + 3 + 1, \\ a_0^{(3)} &= 1, & a_1^{(3)} &= 0, & a_2^{(3)} &= 7, & a_3^{(3)} &= 6, & a_4^{(3)} &= 1, \\ \dim H^3(J - \Theta, \mathbf{C}) &= 15 = 1 + 7 + 6 + 1, \\ a_0^{(4)} &= 1, & a_1^{(4)} &= 0, & a_2^{(4)} &= 15, & a_3^{(4)} &= 25, & a_4^{(4)} &= 10, & a_5^{(4)} &= 1, \\ \dim H^4(J - \Theta, \mathbf{C}) &= 52 = 1 + 15 + 25 + 10 + 1, \\ a_0^{(5)} &= 1, & a_1^{(5)} &= 0, & a_2^{(5)} &= 31, & a_3^{(5)} &= 96, & a_4^{(5)} &= 66, & a_5^{(5)} &= 15, & a_6^{(5)} &= 1, \\ \dim H^5(J - \Theta, \mathbf{C}) &= 210 = 1 + 31 + 96 + 66 + 15 + 1. \end{aligned}$$

5 The character of H^g

In general the character of a graded vector space $H = \bigoplus_n H_n$ is defined by

$$\text{ch } H = \sum_n t^n \dim H_n.$$

By the definition the character of H^g is

$$\text{ch } H^g = \sum_{n=0}^{g+1} a_n^{(g)} t^{n-g}.$$

Proposition 4

$$\begin{aligned} t^g \text{ch } H^g &= 1 - (1-t)^g + (1-t)^{g+1} \left(1 + \left(t \frac{d}{dt} \right)^g (1-t)^{-1} \right) \\ &\quad - \sum_{I_1} (-1)^i \binom{g}{m+i} \binom{g}{m} t^{g+1-m} + \sum_{I_2} (-1)^i \binom{g}{m+i+2} \binom{g}{m} t^{g-m}, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \{(m, i) \mid m+i \leq g-1, 0 \leq m, 1 \leq i\}, \\ I_2 &= \{(m, i) \mid m+i \leq g-2, 0 \leq m, 1 \leq i\}. \end{aligned}$$

Proof. By Proposition 2 we have

$$\begin{aligned}
t^g \text{ch } H^g &= 1 + S'_1 + S'_2 + S'_3, \\
S'_1 &= \sum_{n=2}^{g+1} (-1)^{n-1} \binom{g}{n-2} t^n, \\
S'_2 &= \sum_{n=2}^{g+1} \sum_{i=2}^n (-1)^{n-i} \binom{g+1}{n-i} i^g t^n, \\
S'_3 &= \sum_{n=3}^{g+1} \sum_{i=0}^{n-3} (-1)^i \left(\binom{g}{n-1} \binom{g}{n-2-i} - \binom{g}{n} \binom{g}{n-3-i} \right) t^n.
\end{aligned}$$

It is easy to show that

$$S'_1 = -t^2(1-t)^g + (-1)^g t^{g+2}.$$

By a similar calculation to that of S_2 in (ii) of Proposition 3 we get

$$S'_2 = (1-t)^{g+1} \left(t \frac{d}{dt} \right)^g (1-t)^{-1} - t(1-t)^{g+1} + (-1)^{g+1} t^{g+2}.$$

The sum S'_3 can be written as

$$S'_3 = - \sum_{I_1} (-1)^i \binom{g}{m+i} \binom{g}{m} t^{g+1-m} + \sum_{I_2} (-1)^i \binom{g}{m+i+2} \binom{g}{m} t^{g-m}.$$

The proposition follows from these formulae. ■

6 The space W^k

Let

$$V = \oplus_{i=1}^g \mathbf{C} \alpha_i \oplus_{i=1}^g \mathbf{C} \beta_i = V_- \oplus V_+$$

be a vector space of dimension $2g$. We make $\wedge^p V$, $p \geq 1$ a graded vector space by specifying degree as

$$\begin{aligned}
\deg(\beta_{i_1} \wedge \cdots \wedge \beta_{i_\ell} \wedge \alpha_{j_1} \wedge \cdots \wedge \alpha_{j_k}) &= 1 - k, \quad \ell \geq 1, \\
\deg(\alpha_{j_1} \wedge \cdots \wedge \alpha_{j_k}) &= -k.
\end{aligned}$$

Let $\omega = \sum_{i=1}^g \beta_i \wedge \alpha_i$. Then $\deg \omega = 0$. Define

$$W^k = \frac{\wedge^k V}{\omega \wedge^{k-2} V}.$$

Since $\omega \wedge^{k-2} V$ is a graded subspace of $\wedge^k V$, W^k becomes a graded vector space. Its character is given by

Lemma 8

$$\text{ch } W^k = \binom{g}{k} t^{-k} + \sum_{m=0}^{k-1} \binom{g}{k-m} \binom{g}{m} t^{1-m} - \sum_{m=0}^{k-2} \binom{g}{k-2-m} \binom{g}{m} t^{-m}.$$

Proof. Notice that

$$\text{ch} W^k = \text{ch} \wedge^k V - \text{ch}(\omega \wedge^{k-2} V).$$

According as the decomposition

$$\wedge^k V = \wedge^k V_- \oplus \bigoplus_{m=0}^{k-1} \wedge^{k-m} V_+ \wedge^m V_-.$$

we have

$$\text{ch} \wedge^k V = \binom{g}{k} t^{-k} + \sum_{m=0}^{k-1} \binom{g}{k-m} \binom{g}{m} t^{1-m}.$$

On the other hand we have at least one $d\zeta_i$ in $\omega \wedge^{k-2} V$ and therefore

$$\text{ch}(\omega \wedge^{k-2} V) = \sum_{m=0}^{k-2} \binom{g}{k-2-m} \binom{g}{m} t^{-m}.$$

■

7 Character of affine ring

Let us calculate the character of $\text{gr } A \otimes \wedge^g T^*$. We have, by (13),

$$\begin{aligned} t^g \text{ch}(\text{gr } A \otimes \wedge^g T^*) &= \sum_{n=0}^{\infty} \dim(\text{gr}_n A) t^n \\ &= 1 + \sum_{n=2}^{\infty} (n^g - (n-1)^g) t^n \\ &= (1-t) \left(1 + \left(t \frac{d}{dt} \right)^g (1-t)^{-1} \right). \end{aligned}$$

We introduce a grading in \mathcal{D} by assigning degree as $\deg \partial_i = 1$. Then the character of \mathcal{D} is given by

$$\text{ch } \mathcal{D} = (1-t)^{-g}.$$

Set $H^i = W^i$ for $i < g$. At this point we encounter a remarkable identity.

Corollary 3

$$\text{ch}(gr A \otimes \wedge^g T^*) = \sum_{i=0}^g (-1)^i \text{ch}(\mathcal{D} \otimes H^{g-i}).$$

Proof. By Lemma 8

$$\begin{aligned} \sum_{i=1}^g (-1)^i \text{ch} W^{g-i} &= t^{-g} ((1-t)^g - 1) + \sum_{I_1} (-1)^i \binom{g}{m+i} \binom{g}{m} t^{1-m} \\ &\quad - \sum_{I_2} (-1)^i \binom{g}{m+i+2} \binom{g}{m} t^{-m}, \end{aligned}$$

where I_1, I_2 are the same as those in Proposition 4. Then the corollary follows from Proposition 4. ■

8 Free resolution of affine ring

Define the map $d : \mathcal{D} \otimes \wedge^k V \longrightarrow \mathcal{D} \otimes \wedge^{k+1} V$ by

$$d(P \otimes \eta) = \sum_{i=1}^g \partial_i P \otimes \alpha_i \wedge \eta,$$

and the map $\omega : \mathcal{D} \otimes \wedge^k V \longrightarrow \mathcal{D} \otimes \wedge^{k+2} V$ by

$$\omega(P \otimes \eta) = P \otimes \omega \wedge \eta.$$

Obviously these two maps are \mathcal{D} -linear and commute. Thus d induces a \mathcal{D} -linear map $d : \mathcal{D} \otimes W^k \longrightarrow \mathcal{D} \otimes W^{k+1}$ and defines a complex $(\mathcal{D} \otimes W^\bullet, d)$:

$$0 \longrightarrow \mathcal{D} \longrightarrow \mathcal{D} \otimes W^1 \longrightarrow \cdots \longrightarrow \mathcal{D} \otimes W^g \longrightarrow 0.$$

The following proposition has been proved in [9].

Proposition 5 [9] *The complex $(\mathcal{D} \otimes W^\bullet, d)$ is exact at $\mathcal{D} \otimes W^k$, $k \neq g$.*

Remark It can be easily checked that the map d preserves degree. Notice that the degree in this paper and that in [9] are different.

Let $W^k = \oplus_n W_n^k$ be the homogeneous decomposition. Notice that

$$W_{n-g}^g = \frac{\wedge^{n-1} V_+ \wedge^{g-n+1} V_-}{\omega \wedge^{n-2} V_+ \wedge^{g-n} V_-} \quad n > 1, \quad W_{1-g}^g = 0, \quad W_{-g}^g = \wedge^g V_-.$$

For $J = (j_1, \dots, j_n)$ we set $|J| = n$ and $dz_J = dz_{j_1} \wedge \cdots \wedge dz_{j_n}$ ect.

Lemma 9 *The meromorphic g -form $d\zeta_I \wedge dz_J$, $|I| = n-1 \geq 1$, $|J| = g-n+1$ has a pole on Θ of order at most n and thus can be considered as an element of $gr_n A \otimes \wedge^g T^*$.*

Proof. Let $I = (i_1, \dots, i_{n-1})$ and $J^c = \{1, 2, \dots, g\} \setminus J = \{k_1, \dots, k_{n-1}\}$, $k_1 < \cdots < k_{n-1}$. Then

$$d\zeta_I \wedge dz_J = c \det(\zeta_{i_p k_q})_{1 \leq p, q \leq n-1} dz^{J^c}, \quad (35)$$

where $\zeta_{ij} = \partial_i \zeta_j = \partial_i \partial_j \log \theta(z)$ and $c = \pm 1$. We set $\theta_{i_1 \dots i_l}(z) = \partial_{i_1} \cdots \partial_{i_l} \theta(z)$. Substitutes the expression

$$\zeta_{ij} = -\frac{\theta_i(z)\theta_j(z)}{\theta(z)^2} + \frac{\theta_{ij}(z)}{\theta(z)},$$

into the determinant in (35). Then one easily see that $d\zeta_I \wedge dz_J$ has poles on Θ of order at most n . ■

By Lemma 9 we have a natural map

$$\begin{aligned} \wedge^{n-1} V_+ \wedge^{g-n+1} V_- &\longrightarrow H_{n-g}^g, \\ \beta_I \wedge \alpha_J &\mapsto d\zeta_I \wedge dz_J. \end{aligned} \quad (36)$$

Since $\omega = \sum_{i=1}^g \beta_i \wedge \alpha_i \in V_+ \wedge V_-$ is mapped to

$$\sum_{i,j=1}^g \partial_j \partial_i \log \theta dz_j \wedge dz_i = 0,$$

the map (36) induces a map

$$\varphi_n : W_{n-g}^g \longrightarrow H_{n-g}^g. \quad (37)$$

Lemma 10 *The map φ_n is injective for $n \geq 0$.*

The proof of this lemma is given in the appendix.

In the following we sometimes identify W^g as a graded subspace of H^g by identifying $\beta_I \wedge \alpha_J$ with $d\zeta_I \wedge dz_J$. Let us fix a decomposition

$$H^g = W^g \oplus U^g,$$

where $U^g = \oplus_n U_n^g$ is a graded subspace of H^g . We fix a homogeneous basis $\{u_\beta | \beta \in B\}$ of U^g . With the help of this basis we define the map

$$ev : \mathcal{D} \otimes H^g \longrightarrow \text{gr } A \otimes \wedge^g T^*, \quad (38)$$

in the following way.

Let $\mathcal{D} = \oplus_n \mathcal{D}_n$ be the homogeneous decomposition. For $\alpha_I \wedge \beta_J \in \wedge^g V$, it is possible to write $dz_I \wedge d\zeta_J = f dz^g$ with $f \in A_m$ where $m = |J| + 1$ if $J \neq \phi$, $m = 0$ otherwise by Lemma 9. Define first a map $ev' : \wedge^g V \longrightarrow \text{gr } A \otimes \wedge^g T^*$ by

$$ev'(P \otimes (\alpha_I \wedge \beta_J)) = P(f) dz^g \in \text{gr}_{m+n} A dz^g, \quad P \in \mathcal{D}_n.$$

Since $ev'(\omega \wedge^{g-2} V) = 0$, it induces a well-defined map $ev : \mathcal{D} \otimes W^g \longrightarrow \text{gr } A dz^g$.

Next take $P \otimes u_\beta \in \mathcal{D}_n \otimes U_{m-g}^g$. Write $u_\beta = f_\beta dz^g$ with $f_\beta \in \text{gr}_m A$. Then we set

$$ev(P \otimes u_\beta) = P(f_\beta) dz^g \in \text{gr}_{m+n} A dz^g.$$

Defining a map $d : \mathcal{D} \otimes W^{g-1} \longrightarrow \mathcal{D} \otimes H^g$ by

$$dw = (dw, 0) \in (\mathcal{D} \otimes W^g) \oplus (\mathcal{D} \otimes H^g),$$

where dw in the right hand side is the one defined previously, we get the complex

$$0 \longrightarrow \mathcal{D} \xrightarrow{d} \mathcal{D} \otimes W^1 \xrightarrow{d} \dots \longrightarrow \mathcal{D} \otimes W^{g-1} \xrightarrow{d} \mathcal{D} \otimes H^g \xrightarrow{ev} \text{gr } A \otimes \wedge^g T^* \longrightarrow 0. \quad (39)$$

Theorem 1 *The complex (39) is exact and gives a \mathcal{D} -free resolution of $\text{gr } A \otimes \wedge^g T^*$.*

Proof. By Proposition 5 the complex $(\mathcal{D} \otimes W^\bullet, d)$ is exact at $\mathcal{D} \otimes W^k$, $0 \leq k \leq g-1$. Corollary 1 shows that ev is surjective. By Corollary 3

$$\text{ch}(\text{gr } Adz^g) = \text{ch}(\text{Coker}(\mathcal{D} \otimes W^{g-1} \longrightarrow \mathcal{D} \otimes H^g)).$$

It shows that the complex is exact at $\mathcal{D} \otimes H^g$. This completes the proof. ■

Simply omitting dz^g from $\text{gr } A \otimes \wedge^g T^*$ we get a \mathcal{D} -free resolution of $\text{gr } A$ from (39). It is possible to give a free resolution of A itself using the above result.

Take a representatives of $\{f_\beta dz^g | \beta \in B_{n-g}\}$ from $A_n dz^g$ and denote them by the same letters. The element $d\zeta_I \wedge dz_J$ is considered as an element from Adz^g . We define the map

$$\tilde{ev} : \mathcal{D} \otimes H^g \longrightarrow A,$$

simply by sending $P \otimes f dz^g$ to $P(f) \in A$. Again we have the complex

$$0 \longrightarrow \mathcal{D} \xrightarrow{d} \mathcal{D} \otimes W^1 \xrightarrow{d} \dots \longrightarrow \mathcal{D} \otimes W^{g-1} \xrightarrow{d} \mathcal{D} \otimes H^g \xrightarrow{\tilde{ev}} A \longrightarrow 0, \quad (40)$$

where the maps other than \tilde{ev} are the same as before.

Corollary 4 *The complex (40) is exact and it gives a \mathcal{D} -free resolution of A .*

Proof. The surjectivity of the map $\tilde{e}v$ follows from that of ev . We have to prove $\text{Ker } \tilde{e}v = d(\mathcal{D} \otimes W^{g-1})$. Suppose that $v \in \text{Ker } \tilde{e}v$. Decompose v into homogeneous components as $v = v_n + v_{n-1} + \dots$ with $\deg v_i = i$. Since $\tilde{e}v(v) = 0$, we have $\tilde{e}v(v_n) = -\tilde{e}v(v_{n-1}) - \dots \in A_{n-1+g}$. Thus $ev(v_n) = 0$. Since (39) is exact, $v_n = dw_n$ in $\text{gr } Adz^g$ for some $w_n \in (\mathcal{D} \otimes W^{g-1})_n$. Then $\tilde{e}v(v - dw_n) = \tilde{e}v(v) = 0$. Now $v - dw_n = v_{n-1} + v_{n-2} + \dots$. Thus repeating the same argument we finally have $v = dw$ for some $w \in \mathcal{D} \otimes W^{g-1}$. \blacksquare

9 Theta functions

In this section we interpret the results of the previous section in terms of theta and abelian functions. For $I = (i_1, \dots, i_r)$, $J = (j_1, \dots, j_r)$ we define the element $(I; J)$ of A by

$$(I; J) = \det(\zeta_{i_k j_l})_{1 \leq k, l \leq r}.$$

In particular $(i; j) = \zeta_{ij}$. By the definition $(I; J)$ is anti-symmetric with respect to the permutations of the indices of I and J respectively and satisfy $(I; J) = (J; I)$. These functions arise from the relation

$$d\zeta_I \wedge dz_{J^c} = \text{sgn}(J, J^c)(I; J)dz^g,$$

where $J^c = (j_{r+1}, \dots, j_g)$, $\{j_{r+1}, \dots, j_g\} = \{1, 2, \dots, g\} \setminus J$ and $\text{sgn}(J, J^c)$ is the sign of the permutation $(J, J^c) : (1, 2, \dots, g) \mapsto (J, J^c)$. Let $\tilde{e}v'$ be the composition $\wedge^g V \longrightarrow W^g \xrightarrow{\tilde{e}v} A$, which maps $\beta_I \wedge \alpha_{J^c}$ to $\text{sgn}(J, J^c)(I; J)$. Then $\tilde{e}v'(\omega \wedge^{g-2} V) = 0$. It is equivalent to the relations

$$\sum_{k=1}^{r+2} (-1)^{k-1} (i_1, \dots, i_r, j_k; j_1, \dots, \hat{j}_k, \dots, j_{r+2}) = 0. \quad (41)$$

All \mathbb{C} -linear relations among $(I; J)$'s are generated by these relations. More precisely

Proposition 6 $\text{Ker}(\tilde{e}v') = \omega \wedge^{g-2} V$.

Proof. The proposition follows from Lemma 10 and Corollary 4. \blacksquare

The case $r = 1$ of (41) gives the obvious relations $\zeta_{ij} = \zeta_{ji}$. The case $r = 2$ occurs for $g \geq 4$ and gives

$$(i_1, j_1; j_2, j_3) - (i_1, j_2; j_1, j_3) + (i_1, j_3; j_1, j_2) = 0.$$

Corollary 4 tells that all \mathcal{D} -linear relations among derivatives of $(I; J)$'s are obtained by differentiating the relations

$$\sum_{i=1}^{r+1} (-1)^{k-1} \partial_i (i_1, \dots, i_r; j_1, \dots, \hat{j}_k, \dots, j_{r+1}) = 0, \quad (42)$$

and taking linear combinations of them. The case $r = 1$ of (42) gives the obvious relations

$$\partial_k \zeta_{ij} = \partial_j \zeta_{ik}, \quad j \neq k. \quad (43)$$

The case $r = 2$ gives the relations of the form

$$\partial_m (ij; kl) - \partial_l (ij; km) + \partial_k (ij; lm) = 0. \quad (44)$$

In the following example we set

$$(I; J)_{i_1 \dots i_p} = \partial_{i_1} \cdots \partial_{i_p} (I; J), \quad (I; J)_{1^{r_1} \dots n^{r_n}} = \partial_1^{r_1} \cdots \partial_n^{r_n} (I; J), \\ \zeta_{i_1 \dots i_p} = \partial_{i_1} \cdots \partial_{i_p} \log \theta(z), \quad \zeta_{1^{r_1} \dots n^{r_n}} = \partial_1^{r_1} \cdots \partial_n^{r_n} \log \theta(z).$$

Example The case of $g = 2$.

In this case $H^2 \simeq W^2$ and

$$H^2 = H_{-2}^2 \oplus H_0^2 \oplus H_1^2, \\ H_{-2}^2 = \mathbb{C} dz^2, \quad H_0^2 = \mathbb{C} \zeta_{11} dz^2 \oplus \mathbb{C} \zeta_{12} dz^2 \oplus \mathbb{C} \zeta_{22} dz^2, \quad H_1^2 = \mathbb{C} (12; 12) dz^2.$$

By Theorem 1 $\text{gr } A$ is generated by $H^2(dz^2)^{-1}$ over \mathcal{D} . This means that

$$\text{gr}_0 A = \mathbb{C} 1, \quad \text{gr}_1 A = \{0\}, \quad \text{gr}_2 A = \mathbb{C} \zeta_{11} \oplus \mathbb{C} \zeta_{12} \oplus \mathbb{C} \zeta_{22},$$

and that $\text{gr}_n A$ for $n \geq 3$ is linearly spanned by $\zeta_{i_1 \dots i_n}$ and $(12; 12)_{j_1 \dots j_{n-3}}$. More precisely the following elements form a \mathbb{C} -linear basis of $\text{gr}_n A$:

$$\zeta_{1^r 2^s} \quad (r + s = n), \quad (12; 12)_{1^{r'} 2^{s'}} \quad (r' + s' = n - 3).$$

In fact $\text{gr}_n A$ is the \mathbb{C} -span of these functions as we said above. On the other hand the number of those functions is $2n - 1 = \dim \text{gr}_n A$. Thus they form a basis. This basis for $n \leq 3$ is previously given in [4].

Example The case $g = 3$.

We introduce the lexicographical order on the set \mathbb{Z}^2 . Then

$$H^3 \simeq W^3 \oplus \mathbb{C} v dz^3, \quad v \in \text{gr}_2 A, \\ H_{-3}^3 = \mathbb{C} dz^3, \quad H_{-1}^3 = \oplus_{1 \leq i, j \leq 3} \mathbb{C} \zeta_{ij} dz^3 \oplus \mathbb{C} v dz^3, \\ H_0^3 = \oplus_{1 \leq i \leq j \leq 3, 1 \leq k \leq l \leq 3, (ij) < (kl)} \mathbb{C} (ij; kl) dz^3, \quad H_1^3 = \mathbb{C} (123; 123) dz^3.$$

Here we take a basis $\{v\}$ of U^3 , $U^3 = \mathbb{C} v dz^3$. The relations (42) with $r = 2$ in this case are

$$\partial_1(ij; 23) - \partial_2(ij; 13) + \partial_3(ij; 12) = 0, \quad (ij) = (12), (13), (23).$$

Using these relations functions of the form $P(\partial) \partial_3(12; ij)$ can be eliminated from the \mathcal{D} -linear span of $H^3(dz^3)^{-1}$. Then we get \mathbb{C} -linear basis of $\text{gr}_n A$ as

$$\zeta_{1^{i_1} 2^{i_2} 3^{i_3}} \quad (i_1 + i_2 + i_3 = n), \quad v_{1^{i_1} 2^{i_2} 3^{i_3}} \quad (i_1 + i_2 + i_3 = n - 2), \\ (12; 12)_{1^{i_1} 2^{i_2}}, \quad (12; 13)_{1^{i_1} 2^{i_2}}, \quad (12; 23)_{1^{i_1} 2^{i_2}}, \quad (i_1 + i_2 = n - 3), \\ (13; 13)_{1^{i_1} 2^{i_2} 3^{i_3}}, \quad (13; 23)_{1^{i_1} 2^{i_2} 3^{i_3}}, \quad (23; 23)_{1^{i_1} 2^{i_2} 3^{i_3}} \quad (i_1 + i_2 + i_3 = n - 3), \\ (123; 123)_{1^{i_1} 2^{i_2} 3^{i_3}} \quad (i_1 + i_2 + i_3 = n - 4).$$

This can be proved by checking that the number of the above functions is $n^3 - (n - 1)^3 = \dim \text{gr}_n A$.

It is possible to give an explicit basis of U^3 . To this end we take Klein's sigma function $\sigma(u) = \sigma(u; \lambda)$, $u = (u_1, u_2, u_3)$ of the genus three non-hyperelliptic curve $y^3 - x^4 - \sum_{0 \leq \alpha < 3, 0 \leq \beta < 2} \lambda_{3\alpha+4\beta} x^\alpha y^\beta = 0$ ([1, 2])

as a theta function. We mainly follow the notations in [2] except indices of variables u_i . In this case we define $\Theta = (\sigma(u) = 0)$, $\partial_i = \partial/\partial u_i$, $\zeta_{i_1 \dots i_k} = \partial_{i_1} \dots \partial_{i_k} \log \sigma(u)$ for $i_j \in \{1, 2, 5\}$, $(I; J) = \det(\zeta_{i_p j_q})_{1 \leq p, q \leq r}$ for $I, J \in \{1, 2, 5\}^r$. The affine ring A , in this case, is the space of meromorphic functions on \mathbb{C}^g which have poles only on the zero set of σ and are periodic with respect to the lattice $2\omega_1\mathbb{Z}^3 + 2\omega_2\mathbb{Z}^3$ associated with σ . All arguments in this paper do not depend on a special choice of theta functions and linear coordinates. Thus Theorem 1 holds without any change. Consequently a \mathbb{C} -linear basis of $\text{gr}_n A$ is given by

$$\begin{aligned} & \zeta_{1^{i_1} 2^{i_2} 5^{i_5}} \quad (i_1 + i_2 + i_5 = n), \quad w_{1^{i_1} 2^{i_2} 5^{i_5}} \quad (i_1 + i_2 + i_5 = n - 2), \\ & (12; 12)_{1^{i_1} 2^{i_2}}, \quad (12; 15)_{1^{i_1} 2^{i_2}}, \quad (12; 25)_{1^{i_1} 2^{i_2}}, \quad (i_1 + i_2 = n - 3), \\ & (15; 15)_{1^{i_1} 2^{i_2} 5^{i_5}}, \quad (15; 25)_{1^{i_1} 2^{i_2} 5^{i_5}}, \quad (25; 25)_{1^{i_1} 2^{i_2} 5^{i_5}} \quad (i_1 + i_2 + i_5 = n - 3), \\ & (125; 125)_{1^{i_1} 2^{i_2} 5^{i_5}} \quad (i_1 + i_2 + i_5 = n - 4). \end{aligned}$$

Here one can take

$$w = \frac{D_2^4 \sigma \cdot \sigma}{\sigma(u; \lambda)^2} = 2(\zeta_{2222} + 6\zeta_{22}^2), \quad (45)$$

where D_2^4 is the Hirota derivative defined by

$$D_2^4 \sigma \cdot \sigma = \frac{\partial^4}{\partial y_2^4} \sigma(u + y; \lambda) \sigma(u - y; \lambda)|_{y=0}.$$

This can be easily checked by using the expansion

$$\sigma(u; \lambda) = S(u) + \sum_{n \geq 6} S_n(u), \quad (46)$$

where

$$S(u) = \frac{1}{20}(u_1^5 - 5u_1 u_2^2 + 4u_5)$$

and $S_n(u)$ is a polynomial in u_1, u_2, u_5 of degree n with the degree being defined by $\deg u_i = i$.

Remark It is asserted that (46) holds in [2, 3]. However the proof of [3] contains some mistake. We could not find a proof of it in a literature. Thus, precisely speaking, the expression (45) is derived assuming the expansion (46).

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A Proof of Lemma 6

Consider the exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1) \longrightarrow \text{gr}_1 \mathcal{O} \longrightarrow 0. \quad (47)$$

By Lemma 1 its cohomology sequence gives the isomorphism

$$\beta : H^0(J, \text{gr}_1 \mathcal{O}) \simeq H^1(J, \mathcal{O}) \simeq \bar{T}^*. \quad (48)$$

Lemma 11 (i) The function $\zeta_i(z)$, $1 \leq i \leq g$, can be naturally considered as an element of $H^0(J, \text{gr}_1 \mathcal{O})$ and form its linear basis, where $\zeta_i(z)$ is defined by (28).

(ii) $\beta(\zeta_j(z)) = \pi \sum_{k=1}^g ((\text{Im } \tau)^{-1})_{jk} d\bar{z}_k$.

Proof. Let $p : V \longrightarrow V/\Gamma = J$ be the projection, where $\Gamma = \mathbb{Z}^g + \tau\mathbb{Z}^g$. We shall use the group cohomology description of the cohomology groups of sheaves on J by pulling them back to V . We refer the appendix to §2 in the book [5]. We mainly follow the notations in that book.

For a sheaf \mathcal{F} on J such that $H^i(V, p^* \mathcal{F}) = 0$, $i \geq 1$, there is an isomorphism

$$H^p(\Gamma, H^0(V, p^* \mathcal{F})) \simeq H^p(J, \mathcal{F}), \quad p \geq 0, \quad (49)$$

where the left hand side is the group cohomology with the value in the Γ -module $H^0(V, p^* \mathcal{F})$. Since V is a Stein manifold and p is a local isomorphism, (49) is valid for a coherent \mathcal{O}_J -module \mathcal{F} . Applying (49) we have

$$H^0(J, \text{gr}_1 \mathcal{O}) \simeq H^0(\Gamma, H^0(V, p^* \text{gr}_1 \mathcal{O})) = H^0(V, p^* \text{gr}_1 \mathcal{O})^\Gamma, \quad (50)$$

where $(\)^\Gamma$ denotes the set of Γ -invariants.

We pull back the sequence (47) to V and take the cohomology sequence of it. Then we get

$$0 \longrightarrow H^0(V, \mathcal{O}_V) \longrightarrow H^0(V, \mathcal{O}(p^* \Theta)) \longrightarrow H^0(V, p^* \text{gr}_1 \mathcal{O}) \longrightarrow 0, \quad (51)$$

since $H^i(V, \mathcal{O}_V) = 0$, $i \geq 1$. Thus

$$H^0(V, p^* \text{gr}_1 \mathcal{O}) \simeq \frac{H^0(V, \mathcal{O}(p^* \Theta))}{H^0(V, \mathcal{O}_V)}.$$

Then

$$H^0(J, \text{gr}_1 \mathcal{O}) \simeq \left(\frac{H^0(V, \mathcal{O}(p^* \Theta))}{H^0(V, \mathcal{O}_V)} \right)^\Gamma. \quad (52)$$

Since $\zeta_j(z) \in H^0(V, \mathcal{O}_V(p^* \Theta))$ and

$$\zeta_j(z + m + \tau n) = \zeta_j(z) - 2\pi i n_j, \quad m, n \in \mathbb{Z}^g,$$

$\zeta_j(z)$ can be considered as an element of the right hand side of (52) and therefore of $H^0(J, \text{gr}_1 \mathcal{O})$. We define the element $a_j(m + \tau n, z)$ in the one cocycle $C^1(\Gamma, H^0(V, \mathcal{O}_V))$ by

$$a_j(m + \tau n, z) = -2\pi i n_j. \quad (53)$$

Let δ_1 be the connecting homomorphism

$$\delta_1 : H^0(\Gamma, H^0(V, p^* \text{gr}_1 \mathcal{O})) \longrightarrow H^1(\Gamma, H^0(V, \mathcal{O}_V))$$

of the group cohomology sequence of (51). One can easily check that

$$\delta_1(\zeta_j(z)) = a_j(m + \tau n, z). \quad (54)$$

Let us calculate the description of $a_j(m + \tau n, z)$ in terms of Dolbeaut cohomology description.

Let $\mathcal{C}_X^{p,q}$ be the sheaf of germs of $C^\infty(p, q)$ -forms on X . Notice that

$$H^1(\Gamma, H^0(V, \mathcal{O}_V)) \simeq H^1(J, \mathcal{O}_J) \simeq \frac{\left(\text{Ker} \left(\bar{\partial} : H^0(V, \mathcal{C}_V^{0,1}) \longrightarrow H^0(V, \mathcal{C}_V^{0,2}) \right) \right)^\Gamma}{\bar{\partial} \left(H^0(V, \mathcal{C}_V^{0,0})^\Gamma \right)}. \quad (55)$$

This isomorphism is given in the following way.

We have a natural map

$$H^1(\Gamma, H^0(V, \mathcal{O}_V)) \longrightarrow H^1\left(\Gamma, H^0(V, \mathcal{C}_V^{0,0})\right) \simeq H^1(J, \mathcal{C}_J^{0,0}) = 0.$$

This means that any one cocycle $f(\gamma, z) \in C^1(\Gamma, H^0(V, \mathcal{O}_V))$ is written as

$$f(\gamma, z) = h(z + \gamma) - h(z),$$

for some C^∞ function $h(z)$ on V . One can easily check that the isomorphism (55) is given by

$$f(\gamma, z) \mapsto \bar{\partial}h(z).$$

For the one cocycle $a_j(\gamma, z)$ we have

$$\begin{aligned} a_j(\gamma, z) &= h_j(z + \gamma) - h_j(z), \quad \gamma \in \Gamma, \\ h_j(z) &= \pi \sum_{k=1}^g ((Im \tau)^{-1})_{jk} (\bar{z}_k - z_k). \end{aligned}$$

Thus the isomorphism (55) is given by

$$a_j(\gamma, z) \mapsto \pi \sum_{k=1}^g ((Im \tau)^{-1})_{jk} d\bar{z}_k. \quad (56)$$

This proves (ii) of Lemma 11. Since β is an isomorphism and $\beta(\zeta_j(z))$, $1 \leq j \leq g$ are linearly independent due to (56), (i) of Lemma 11 is also proved. \blacksquare

By Lemma 11 and (4) we have

$$\Xi = d \log \theta = \sum_{j=1}^g \zeta_j(z) dz_j \in H^0(J, \text{gr}_1 \mathcal{O}) \otimes H^0(J, \Omega^1) \simeq H^0(J, \text{gr}_1 \Omega^1).$$

Consider the connecting homomorphism

$$\tilde{\beta} : H^0(J, \text{gr}_1 \Omega^1) \simeq H^1(J, \Omega^1) \simeq \bar{T}^* \otimes T^*. \quad (57)$$

of the cohomology sequence of

$$0 \longrightarrow \Omega^1 \longrightarrow \Omega^1(1) \longrightarrow \text{gr}_1 \Omega^1 \longrightarrow 0.$$

By Lemma 11 we have

$$\tilde{\beta}(\Xi) = \sum_{j=1}^g \beta(\zeta_j(z)) \wedge dz_j = \pi \sum_{i,j=1}^g ((Im \tau)^{-1})_{ij} d\bar{z}_i \wedge dz_j = \hat{\omega} \quad (58)$$

where we use the fact that τ is symmetric.

Consider the composition of maps

$$\Omega^{g-n} \longrightarrow \text{gr}_0 \Omega^{g-n} \xrightarrow{\Xi \wedge} \Xi \wedge \text{gr}_0 \Omega^{g-n} \hookrightarrow \text{gr}_1 \Omega^{g-n+1},$$

where the first map is the natural projection. It induces maps of cohomologies

$$H^i(J, \Omega^{g-n}) \simeq H^i(J, \text{gr}_0 \Omega^{g-n}) \simeq H^i(J, \Xi \wedge \text{gr}_0 \Omega^{g-n}) \longrightarrow H^i(J, \text{gr}_1 \Omega^{g-n+1}), \quad i \leq n-2, \quad (59)$$

where we use Lemma 2 and Lemma 5. Let us study the map (59). In sum the map (59) is induced from the map

$$\Omega^{g-n} \xrightarrow{\Xi \wedge} \text{gr}_1 \Omega^{g-n+1}. \quad (60)$$

Thus we consider more generally the bilinear map

$$\Omega^{g-n} \times \text{gr}_1 \Omega^1 \longrightarrow \text{gr}_1 \Omega^{g-n+1}. \quad (61)$$

We pull it back to V and get the map of Γ -modules

$$H^0(V, \Omega_V^{g-n}) \times H^0(V, p^* \text{gr}_1 \Omega^1) \longrightarrow H^0(V, p^* \text{gr}_1 \Omega^{g-n+1}). \quad (62)$$

It induces a cup product on cohomologies ([5])

$$H^i(\Gamma, H^0(V, \Omega_V^{g-n})) \times H^j(\Gamma, H^0(V, p^* \text{gr}_1 \Omega^1)) \longrightarrow H^{i+j}(\Gamma, H^0(V, p^* \text{gr}_1 \Omega^{g-n+1})). \quad (63)$$

Taking $j = 0$ we have a map

$$H^i(\Gamma, H^0(V, \Omega_V^{g-n})) \times H^0(\Gamma, H^0(V, p^* \text{gr}_1 \Omega^1)) \longrightarrow H^i(\Gamma, H^0(V, p^* \text{gr}_1 \Omega^{g-n+1})). \quad (64)$$

Then $\Xi \in H^0(\Gamma, H^0(V, p^* \text{gr}_1 \Omega^1))$ defines a map

$$H^i(\Gamma, H^0(V, \Omega_V^{g-n})) \longrightarrow H^i(\Gamma, H^0(V, p^* \text{gr}_1 \Omega^{g-n+1})), \quad (65)$$

which, by the definition, coincides with the map (59). Let us describe this map in terms of Dolbeaut cohomology. To this end consider the diagram

$$\begin{array}{ccccc} H^i(J, \Omega^{g-n}) & \times & H^0(J, \text{gr}_1 \Omega^1) & \longrightarrow & H^i(J, \text{gr}_1 \Omega^{g-n+1}) \\ & \downarrow 1 \times \tilde{\beta} & & & \downarrow \delta_2 \\ H^i(J, \Omega^{g-n}) & \times & H^1(J, \Omega^1) & \longrightarrow & H^{i+1}(J, \Omega^{g-n+1}). \end{array} \quad (66)$$

Here the down arrows are isomorphisms, δ_2 is the connecting homomorphism of the long cohomology exact sequence of

$$0 \longrightarrow \Omega^{g-n+1} \longrightarrow \Omega^{g-n+1}(1) \longrightarrow \text{gr}_1 \Omega^{g-n+1} \longrightarrow 0,$$

and the horizontal map in the second row is the cup product of cohomologies. By a direct calculation in terms of group cohomologies one can show that (66) is a commutative diagram. The cup product of group cohomologies is compatible with that of sheaf cohomologies. In Dolbeaut description the cup product is given by the exterior product. Therefore, by (58), the map α is given by wedging $\hat{\omega}$. \blacksquare

B Proof of (30)

Lemma 12 *As an element of $H^1(J, \mathbb{C})$ we have*

$$d\zeta_j = \pi \sum_{k=1}^g ((Im \tau)^{-1})_{jk} (d\bar{z}_k - dz_k). \quad (67)$$

Proof. We denote the one cycle on J specified by the element $m + \tau n$ in Γ by $\gamma(m, n)$. Then

$$\int_{\gamma(m, n)} d\zeta_j = -2\pi i n_j, \quad \int_{\gamma(m, n)} (d\bar{z}_k - dz_k) = -2i \sum_{j=1}^g \text{Im } \tau_{kj} n_j.$$

The lemma follows from this. ■

By Lemma 12 we have

$$\begin{aligned} \omega = \sum_{j=1}^g d\zeta_j \wedge dz_j &= \pi \sum_{j,k=1}^g ((\text{Im } \tau)^{-1})_{jk} (d\bar{z}_k - dz_k) \wedge dz_j \\ &= \pi \sum_{j,k=1}^g ((\text{Im } \tau)^{-1})_{jk} d\bar{z}_k \wedge dz_j \\ &= \hat{\omega}, \end{aligned}$$

where we use the symmetry of $\text{Im } \tau$. ■

C Proof of Lemma 10

For $n = 0, 1, g + 1$ the lemma is obvious. We assume $2 \leq n \leq g$. By Lemma 2, Lemma 3 (iii) and Lemma 11 (i)

$$\bar{T}^* \simeq H^1(J, \mathcal{O}) \xrightarrow{\delta_1^{-1}} H^0(J, \text{gr}_1 \mathcal{O}) \xrightarrow{d} H^0(J, d\text{gr}_1 \mathcal{O}) = \oplus_{i=1}^g \mathbb{C} d\zeta_i \simeq V_+, \quad (68)$$

where δ_1 is the connecting homomorphism of the cohomology sequence of (47) and the last isomorphism is given by $d\zeta_j \mapsto \beta_j$. On the other hand $T^* = \oplus_{i=1}^g \mathbb{C} dz_i \simeq V_-$ by the map $dz_j \mapsto \alpha_j$. Thus we have, by (25) and (26),

$$H^i(J, \Xi \wedge \text{gr}_0 \Omega^{g-n}) \simeq \wedge^i V_+ \wedge^{g-n} V_- \quad i \leq n-2, \quad (69)$$

$$H^i(J, \text{gr}_1 \Omega^{g-n+1}) \simeq \wedge^{i+1} V_+ \wedge^{g-n+1} V_- \quad i \geq 0. \quad (70)$$

By Lemma 6, Lemma 11 and (68) the map

$$\alpha : H^i(J, \Xi \wedge \text{gr}_0 \Omega^{g-n}) \longrightarrow H^i(J, \text{gr}_1 \Omega^{g-n+1})$$

is given, in the description of (69) and (70), by wedging $\omega = \sum_{j=1}^g \beta_j \wedge \alpha_j$.

Consider the exact sequence (32). For $n = 2$ we have the exact sequence

$$0 \longrightarrow \wedge^{g-2} V_- \xrightarrow{\wedge \omega} V_+ \wedge^{g-1} V_- \longrightarrow H^0(J, d\text{gr}_1 \Omega^{g-1}) \longrightarrow \dots \quad (71)$$

Since

$$d\text{gr}_1 \Omega^{g-1} \simeq \text{gr}_2 \Omega^g, \quad H^0(J, \text{gr}_2 \Omega^g) \simeq H_{2-g}^g, \quad (72)$$

we get the injective map

$$W_{2-g}^g = \frac{V_+ \wedge^{g-1} V_-}{\omega \wedge^{g-2} V_-} \longrightarrow H_{2-g}^g. \quad (73)$$

This map obviously coincides with φ_2 . Thus the $n = 2$ case of the lemma is proved.

Let us assume $n \geq 3$. By the isomorphisms (22) with $k = n - 1$ and (11) we have

$$H^{n-2}(J, \text{dgr}_1 \Omega^{g-n+1}) \simeq H^1(J, \text{dgr}_{n-2} \Omega^{g-2}) \simeq H_{n-g}^g, \quad n \geq 3. \quad (74)$$

Thus we have the injective map

$$\psi_n : W_{n-g}^g \longrightarrow H_{n-g}^g. \quad (75)$$

Let us prove that ψ_n coincides with φ_n . To this end we explicitly describe the isomorphism (74). We use the notations in Appendix A.

Lemma 13 *Let I, J be the index sets such that $|I| = n \geq 1$, $|J| = l$, $n + l \leq g$. Write $I = (i_1, \dots, i_n)$. Then*

$$\begin{aligned} d\zeta_I \wedge dz_J &= d(\eta \wedge dz_J), \\ \eta &= \frac{1}{n} \sum_{k=1}^n (-1)^{k-1} \zeta_{i_k} d\zeta_{i_1} \wedge \dots \wedge \widehat{d\zeta_{i_k}} \wedge \dots \wedge d\zeta_{i_n}, \end{aligned} \quad (76)$$

and $\eta \in H^0(V, \Omega_V^{n-1}(np^*\Theta))$. In particular

$$d\zeta_I \wedge dz_J \in H^0(V, d\Omega_V^{n+l-1}(np^*\Theta)).$$

Proof. Notice that the determinant

$$\begin{vmatrix} \zeta_{i_1} & \zeta_{i_1 r_1} & \cdots & \zeta_{i_1 r_{n-1}} \\ \vdots & \vdots & & \vdots \\ \zeta_{i_n} & \zeta_{i_n r_1} & \cdots & \zeta_{i_n r_{n-1}} \end{vmatrix}$$

has poles on $p^*\Theta$ of order at most n for any (r_1, \dots, r_{n-1}) . The lemma follows from this. \blacksquare

Let $a_j(\gamma)$ be the one cocycle defined in (53) and

$$f_{IJ}(\gamma_1, \dots, \gamma_k | z) = a_{r_1}(\gamma_1) \cdots a_{r_k}(\gamma_k) d\zeta_I \wedge dz_J, \quad (77)$$

where r_1, \dots, r_k are distinct. We omit writing r'_j 's in f_{IJ} for the sake of simplicity. By Lemma 13, for I, J satisfying $|I| = n - k - 1$, $|J| = g - n + 1$, $f_{IJ}(\gamma_1, \dots, \gamma_k | z)$ can be considered as an element of

$$H^k(\Gamma, H^0(V, p^* \text{dgr}_{n-k-1} \Omega^{g-k-1})) \simeq H^k(J, \text{dgr}_{n-k-1} \Omega^{g-k-1}). \quad (78)$$

Let ι_k be the connecting homomorphism (18)

$$\iota_k : H^k(\Gamma, H^0(V, p^* \text{dgr}_{n-k-1} \Omega^{g-k-1})) \simeq H^{k+1}(\Gamma, H^0(V, p^* \text{dgr}_{n-k-2} \Omega^{g-k-2})). \quad (79)$$

Then, using Lemma 13, we have

$$\begin{aligned} \iota_k(f_{IJ})(\gamma_0, \dots, \gamma_k) &= \\ a_{r_1}(\gamma_1) \cdots a_{r_k}(\gamma_k) \frac{1}{n-k-1} \sum_{l=1}^{n-k-1} (-1)^{l-1} a_{i_l}(\gamma_0) d\zeta_{i_1} \wedge \dots \wedge \widehat{d\zeta_{i_l}} \wedge \dots \wedge d\zeta_{i_{n-k-1}} \wedge dz_J. \end{aligned} \quad (80)$$

With the help of (80) we find that the composition of maps

$$H^0(J, \text{gr}_n \Omega^g) \simeq H^0(J, \text{dgr}_{n-1} \Omega^{g-1}) \longrightarrow H^1(J, \text{dgr}_{n-2} \Omega^{g-2}) \simeq H^{n-2}(J, \text{dgr}_1 \Omega^{g-n+1})$$

is given by

$$d\zeta_I \wedge dz_J \mapsto \frac{1}{n-1} \sum_{k=1}^{n-1} (-1)^{n-1-k} a_{I \setminus \{i_k\}}(\gamma_{n-2}, \dots, \gamma_1) d\zeta_{i_k} \wedge dz_J, \quad (81)$$

where $I = (i_1, \dots, i_{n-1})$, $|J| = g - n + 1$, and

$$a_{I \setminus \{i_k\}}(\gamma_{n-2}, \dots, \gamma_1) = a_{i_1}(\gamma_1) \cdots a_{i_{k-1}}(\gamma_{k-1}) a_{i_{k+1}}(\gamma_k) \cdots a_{i_{n-1}}(\gamma_{n-2}).$$

On the other hand the map

$$\wedge^{n-1} H^1(J, \mathcal{O}) \otimes H^0(J, \Omega^{g-n+1}) \simeq H^{n-2}(J, \text{gr} \Omega^{g-n+1})$$

is given by

$$\zeta_I dz_J \mapsto \frac{1}{n-1} \sum_{k=1}^{n-1} (-1)^{n-1-k} a_{I \setminus \{i_k\}}(\gamma_{n-2}, \dots, \gamma_1) \zeta_{i_k} \wedge dz_J$$

and consequently the composition of maps

$$\wedge^{n-1} V_+ \wedge^{g-n+1} V_- \simeq \wedge^{n-1} H^1(J, \mathcal{O}) \otimes H^0(J, \Omega^{g-n+1}) \simeq H^{n-2}(J, \text{gr} \Omega^{g-n+1}) \longrightarrow H^{n-2}(J, \text{dgr}_1 \Omega^{g-n+1})$$

is given by

$$\beta_I \wedge \alpha_J \mapsto \frac{1}{n-1} \sum_{k=1}^{n-1} (-1)^{n-1-k} a_{I \setminus \{i_k\}}(\gamma_{n-2}, \dots, \gamma_1) d\zeta_{i_k} \wedge dz_J. \quad (82)$$

Comparing (81) with (82) we have $\psi_n = \varphi_n$. ■

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